

Contents

Chapter 1	Chaos for Nearly Integrable Systems	1
1.1	Direct methods of perturbation theory for solitons	1
1.2	Perturbation theory based on the inverse scattering transform	4
1.3	Motion of a soliton in a driven Sine-Gordon equation	8
1.3.1	Soliton motion of Sine-Gordon equation	8
1.3.2	Motion of a SG soliton in the fields of two waves	10
1.3.3	Stochastic dynamics of a three-dimensional bubble in a driven SG equation	11
1.3.4	SG soliton similar to the Fermi-Pasta-Ulam problem	13
1.3.5	Dynamical chaos of a breather under the action of an external field	14
1.3.6	Dynamical chaos in the SG system with parametric excitation	16
1.3.7	Stochastization of soliton lattices in the perturbed SG equation	18
1.4	Motion of the soliton of nonlinear Schrödinger equation with damping under the action of an external field	20
1.4.1	Nonlinear Schrödinger equation	20
1.4.2	Stochastic dynamics of NLS solitons in a periodic potential	20
1.5	Dynamical chaos of the KdV equation and the perturbation equations	23
1.5.1	Chaotic state of the cnoidal wave in the periodic inhomogeneous medium	23
1.5.2	Karamoto-Sivashinsky equation	24
Chapter 2	Some Numerical Results and Their Analysis	26
2.1	Coherent structure and numerical calculation results	27
2.2	Fundamental analysis	54
2.2.1	Connections between NLS equation and Sine-Gordon equation	54
2.2.2	Space independent fixed point	55
2.2.3	Space dependent fixed point	57
2.2.4	Integrable structure of nonlinear Schrödinger equation	59
2.2.5	The Whisker ring of focusing nonlinear Schrödinger equation	75
Chapter 3	Homoclinic Orbits in a Four Dimensional Model of a Perturbed Nonlinear Schrödinger Equation	91
3.1	Dynamics and geometric structure for the unperturbed system	91

3.1.1	\mathcal{M}_0 and $W^s(\mathcal{M}_0) \cap W^u(\mathcal{M}_0)$	93
3.1.2	The dynamics on \mathcal{M}_0	95
3.1.3	The unperturbed homoclinic orbits and their relationship to the dynamics on \mathcal{M}_0 and $W^s(\mathcal{M}_0) \cap W^u(\mathcal{M}_0)$	95
3.2	Geometric structure of the perturbed system	98
3.2.1	The persistence of $\mathcal{M}_0, W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$ under perturbation	99
3.2.2	The dynamics on \mathcal{M}_ε near resonance	99
3.3	Fiber representations of stable and unstable manifolds	103
3.3.1	Representation of $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$ through homoclinic orbits	103
3.3.2	An intuitive introduction to fibrations of stable and unstable manifolds	104
3.3.3	A second example	107
3.3.4	Fibers for $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$ of the two mode equations	111
3.3.5	Properties and characteristics of the fibers	112
3.3.6	Fibers representations for the subset of $W^u(q_\varepsilon)$ and $W_{\text{loc}}^s(\mathcal{A} \subset \mathcal{M}_\varepsilon)$	113
3.4	Homoclinic orbits for q_ε	114
3.4.1	Homoclinic coordinates and the hyperplane Σ	115
3.4.2	The Melnikov function for $W^s(\mathcal{A} \subset \mathcal{M}_\varepsilon) \cap W^u(q_\varepsilon)$	117
3.4.3	Explicit expression of the Melnikov function at $I = 1$	121
3.4.4	The existence of orbits homoclinic to q_ε	124
3.5	Numerical results of orbits homoclinic to q_ε	130
3.5.1	Numerical computation for periodic solution	130
3.5.2	Computation for homoclinic manifolds	131
3.6	The dynamical consequences of orbits homoclinic to q_ε : the existence and property of chaos	137
3.6.1	Construction of the domains for the maps	139
3.6.2	Construction of the map P_0 near the origin	140
3.6.3	Construction of the map along the homoclinic orbits outside a neighborhood of the origin	143
3.6.4	The full map, $P \equiv P_0 \circ P_1 : \Pi_0 \rightarrow \Pi_0$	145
3.6.5	Verification of the hypotheses of the theorem for the two-mode truncation	146
Chapter 4 Homoclinic Orbits of a Damped and Forced Sine-Gordon Equation		150
4.1	Structure of the unperturbed system	151
4.1.1	The normally hyperbolic invariant manifold \mathcal{M}	151

4.1.2 The dynamics on \mathcal{M} 152

4.1.3 $W^s(\mathcal{M}), W^u(\mathcal{M})$ and the homoclinic manifold 152

4.1.4 The dynamics on Γ and its relation to the dynamics in \mathcal{M} 153

4.2 Structure of the perturbed system 154

4.2.1 The persistence of $\mathcal{M}, W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ under perturbation 154

4.2.2 The dynamics on \mathcal{M}_ε 156

4.2.3 The fibering of $W^s(\mathcal{A}_\varepsilon)$ and $W^u(\mathcal{A}_\varepsilon)$:
the singular perturbation nature 160

4.3 The existence of a homoclinic connection to p_ε 163

4.3.1 $W^u(p_\varepsilon) \subset W^s(\mathcal{A}_\varepsilon)$: The higher dimensional Melnikov theory 164

4.3.2 $W^u(p_\varepsilon) \cap W^s(p_\varepsilon)$: a homoclinic orbit to p_ε 166

4.4 Chaos: Silnikov’s theorem 170

4.5 An application: model dynamics of the damped, driven,
nonlinear Schrödinger equation 171

4.5.1 The unperturbed integrable structure 173

4.5.2 Dynamics near the resonance on \mathcal{A}_ε 178

4.5.3 Calculation of the Melnikov function 180

4.5.4 The existence of an orbit homoclinic to p_ε 183

4.5.5 The geometrical interpretation of chaos in phase space 185

**Chapter 5 Persistent Homoclinic Orbits for a Perturbed
Nonlinear Schrödinger Equation** 189

5.1 Introduction 189

5.2 Analysis of space-independent solutions and
motion on the invariant plane 190

5.2.1 Motion on the invariant plane 190

5.2.2 The stable manifolds at Q in Π_c 192

5.3 The equations in a neighborhood of the circle of fixed points 197

5.3.1 Basic equations 197

5.3.2 Normal forms 200

5.3.3 Local equations 205

5.4 Theory of invariant manifolds 206

5.4.1 Existence of local invariant manifolds 206

5.4.2 The fibration for invariant manifolds 216

5.4.3 Stable manifold to Q in M_ε 224

5.5 Global integrable theory 231

5.5.1 Lax pair 231

5.5.2	Zakharov-Shabat spectral problem	231
5.5.3	The basic example	234
5.5.4	Homoclinic orbits and whiskered tori	235
5.5.5	An important invariant	239
5.5.6	$F'(q_h)$	241
5.6	Persistent homoclinic orbit	242
5.6.1	The first measurement	243
5.6.2	The second measurement	250
5.6.3	Existence of a homoclinic orbit	254
Chapter 6 Homoclinic Orbits and Chaos for the Discrete Disturbed Nonlinear Schrödinger Equation		257
6.1	Integrable case	257
6.1.1	Spectral theory of L_n	259
6.1.2	Hyperbolic structure and homoclinic orbits	260
6.2	Persistent invariant manifolds	263
6.2.1	Persistent invariant plane	264
6.2.2	Persistent invariant manifold theorem	265
6.2.3	The proof of the local persistent invariant manifold theorem	267
6.3	Fenichel fibers	272
6.3.1	An example showing fenichel fibers	272
6.3.2	Fiber theorem	273
6.3.3	The unique explicit fenichel fiber for “figure 8 \otimes A”	275
6.4	Melnikov measurement: $W^u(q_\varepsilon) \cap W_\varepsilon^{cs}$	276
6.4.1	Main argument	276
6.4.2	Derivation of Melnikov integral	280
6.4.3	Approximation	290
6.4.4	Computation for $\hat{M}_{\hat{F}_1}$	292
6.4.5	The intersection between $W^u(q_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon) \subset W_\varepsilon^{cs}$	294
6.5	Existence of orbits homoclinic to q_ε : the second measurement	295
6.6	General theory of symbolic dynamics	302
6.6.1	General framework	302
6.6.2	Smooth normal form reduction	304
6.6.3	Some definitions	305
6.6.4	Poincaré map P_0^1	310
6.6.5	Poincaré map P_1^0	310
6.6.6	Fixed point of Poincaré map $P \equiv P_1^0 \circ P_0^1$	313

6.6.7	Smale horseshoes	322
6.6.8	Symbol dynamics	333
6.7	Application to discrete NLS systems	337
6.7.1	Transformation of (6.6.1) to the form (6.1.3)	337
6.7.2	The Generic assumptions	338
6.7.3	Smale horseshoes and chaos created by a pair of homoclinic orbits in the discrete nonlinear Schrödinger systems	339
Chapter 7 Persistent Homoclinic Orbits for the Perturbed Sine-Gordon Equation 348		
7.1	Persistent homoclinic orbits for a kind of Sine-Gordon equation under dissipative perurbation	348
7.2	Persistent homoclinic orbits for another kind of Sine-Gordon equation under dissipative perturbation	355
7.3	Persistent homoclinic orbits for a kind of Klein-Gordon equation under small perturbation	373
Chapter 8 Persistent Homoclinic Orbits of Perturbed High-order Nonlinear Schrödinger Equations 381		
8.1	Persistent homoclinic orbits of a perturbed cubic-quintic NLS equation	381
8.1.1	Some fundamental results	381
8.1.2	The equations in a neighborhood of C_ω	387
8.1.3	Invariant manifolds	390
8.1.4	Persistent homoclinic orbit	399
8.2	Homoclinic orbits in a six dimensional model of derivative nonlinear Schrödinger equation	405
8.2.1	The Fourier truncation of a perturbed derivative NLS equation	406
8.2.2	Persistence of the normally hyperbolic invariant manifold	413
8.2.3	Persistence of the homoclinic orbits	416
8.3	Persistent homoclinic orbits for a perturbed coupled nonlinear Schrödinger system	420
8.3.1	The preliminary results	420
8.3.2	An equation in a neighborhood of S_ω	426
8.3.3	Existence of local invariant manifolds	432
8.3.4	Homoclinic orbit of unperturbed system	441
8.3.5	Persistent homoclinic orbit	442
8.4	Persistent homoclinic orbits for a perturbed nonlinear Schrödinger	

equation with derivation term under a small perturbation ·····	448
8.4.1 The preliminary results ·····	448
8.4.2 Analysis of space-independent solutions ·····	449
8.4.3 Equation in a neighborhood of C_ω ·····	452
8.4.4 Invariant manifolds ·····	454
8.4.5 Persistent homoclinic orbit ·····	461
Chapter 9 Homoclinic Orbits of a Perturbed	
Nonlinear Schrödinger Equation ·····	468
9.1 Main theorems and establishment of basic equations ·····	468
9.2 Invariant manifolds and invariant foliations ·····	472
9.3 Homoclinic orbits ·····	509
9.3.1 Homoclinic orbits for unperturbed NLS ·····	510
9.3.2 The first measurement ·····	512
9.3.3 Second measurement ·····	519
9.3.4 Existence of a Homoclinic Orbit ·····	522
Chapter 10 Morse Functions and Floquet Theory ·····	525
10.1 Morse and Melnikov functions for nonlinear Schrödinger equation ···	525
10.1.1 Floquet Spectral Theory ·····	526
10.1.2 Critical Structure of F_j ·····	532
10.1.3 The Morse description of the isospectral stratification ·····	536
10.1.4 A Melnikov vector ·····	547
10.2 Hill equation ·····	548
10.3 Topological classification of integrable partial differential equations ···	556
Bibliography ·····	561